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THE DIFFERENCE MATRICES OF GENERALIZED FIBONACCI AND LUCAS SEQUENCES

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ABSTRACT. The difference sequence $\Delta\mathbf{a} = (\Delta a_0, \Delta a_1, \Delta a_2, \ldots)^T$ of a sequence $\mathbf{a} = (a_0, a_1, a_2, \ldots)^T$ is defined by $\Delta a_i = a_{i+1} - a_i$ for each $i = 0, 1, 2, \ldots$. Let $\Delta^i\mathbf{a} = (\Delta^i a_0, \Delta^i a_1, \Delta^i a_2, \ldots)^T$, $i = 0, 1, 2, \ldots$, be the ith difference sequence defined inductively by $\Delta^i\mathbf{a} = \Delta(\Delta^{i-1}\mathbf{a})$ where $\Delta^0\mathbf{a} = \mathbf{a}$. Let $\mathbb{M}^{\mathbf{a}} = [m_{ij}], (i, j = 0, 1, 2, \ldots)$ denote the difference matrix of $\mathbf{a}$ whose ith row is the transpose of the ith difference sequence and let $\mathbb{H}^{\mathbf{a}}$ denote an infinite Hankel matrix with $\mathbf{a}^T$ as its first row. The entries in $\mathbb{M}^{\mathbf{a}}$ satisfy the Γ-law defined by the recurrence relation $m_{i,j+1} = m_{i+1,j} + m_{ij}$. In this paper, the Γ-law is applied to examine the recurrence relations of difference sequences of $\mathbf{F}_{k,l}$ and $\mathbf{L}_{k,l}$, where $\mathbf{F}_{k,l}$ and $\mathbf{L}_{k,l}$ are (k,l)-Fibonacci and (k,l)-Lucas sequences, respectively, for positive real numbers k and l. Finally, we show that $\mathbb{M}^{\mathbf{F}_{k,l}} = P^{-1}\mathbb{H}^{\mathbf{F}_{k,l}}$ and $\mathbb{M}^{\mathbf{L}_{k,l}} = P^{-1}\mathbb{H}^{\mathbf{L}_{k,l}}$, where P is the Pascal matrix.

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1. INTRODUCTION

Fibonacci and Lucas sequences with the Golden Mean, $\tau = \frac{1+\sqrt{5}}{2}$, have long been of great interest in a wide range of fields, from Architecture to, Nature, Art, and high energy particle physics, as well as in mathematics, such as in number theory and combinatorics [5]. The sequence $\mathbf{F} = (F_0, F_1, F_2, \ldots)^T$ defined by

$$F_0 = 1,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}\ (n \geq 2)$$

is called the Fibonacci sequence and the sequence $\mathbf{L} = (L_0, L_1, L_2, \ldots)^T$ defined by

$$L_0 = 2,\ L_1 = 1,\ L_n = L_{n-1} + L_{n-2}\ (n \geq 2)$$

is called the Lucas sequence [1, 4]. Various generalizations of the Fibonacci and Lucas sequences have been introduced [3, 5, 6, 10]. For example, for each positive real number k, the sequence $\mathbf{F}_k = (F_{k,0}, F_{k,1}, F_{k,2}, \ldots)^T$ defined by

$$F_{k,0} = 0,\ F_{k,1} = 1,\ F_{k,n} = kF_{k,n-1} + F_{k,n-2}\ (n \geq 2), \tag{1}$$

is the k-Fibonacci sequence [5]. By applying a difference relation to the sequence, Falcon [6] introduced another generalized Fibonacci sequences as follows.

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Definition 1.1. *For each positive real number k, let $\mathbf{F}_k$ denote the k-Fibonacci sequence and let $\Delta F_{k,n} = F_{k,n+1} - F_{k,n}$ for a nonnegative integer n. The sequence*

$$\Delta^i \mathbf{F}_k = (\Delta^i F_{k,0}, \Delta^i F_{k,1}, \Delta^i F_{k,2}, \ldots)^T,$$

defined inductively by $\Delta^i \mathbf{F}_k = \Delta(\Delta^{i-1}\mathbf{F}_k)$, where $\Delta^0 \mathbf{F}_k = \mathbf{F}_k$, is called the ith difference sequence of $\mathbf{F}_k$ for $i \geq 0$. For brevity, $\Delta^i \mathbf{F}_k$ and $\Delta^i F_{k,n}$ will be denoted by $\mathbf{F}_k^{(i)}$ and $F_{k,n}^{(i)}$, respectively. The difference sequences of $\mathbf{L}_k$ can be defined similarly.

Falcon [6] presented several formulae of the difference sequences of $\mathbf{F}_k$, including:

- $F_{k,n+1}^{(i)} = kF_{k,n}^{(i)} + F_{k,n-1}^{(i)}$.
- $F_{k,n}^{(i)} = (k-1)F_{k,n}^{(i-1)} + F_{k,n-1}^{(i-1)}$.
- $\sum_{j=0}^{n} F_{k,j}^{(i)} = F_{k,n+1}^{(i-1)} - F_{k,0}^{(i-1)}$.

Furthermore, he investigated the first terms of the difference sequences of $\mathbf{F}_k$, which are applied to provide the k-Fibonacci Newton polynomial interpolation with its generating functions. Motivated by (1) and Definition 1.1, we consider generalizations of k-Fibonacci (Lucas) and the difference sequences of k-Fibonacci (Lucas) sequences. Spinadel introduced these generalizations [11] as a secondary Fibonacci sequence.

Definition 1.2. *For a pair of positive real numbers k and l, the sequence*

$$\mathbf{F}_{k,l} = (F_{(k,l),0}, F_{(k,l),1}, F_{(k,l),2}, \ldots)^T$$

is called the (k,l)-Fibonacci sequence defined by

$$F_{(k,l),0} = 0,\ F_{(k,l),1} = 1,\ F_{(k,l),n} = kF_{(k,l),n-1} + lF_{(k,l),n-2}\ (n \geq 2) \tag{2}$$

and the sequence

$$\mathbf{L}_{k,l} = (L_{(k,l),0}, L_{(k,l),1}, L_{(k,l),2}, \ldots)^T$$

is called the (k,l)-Lucas sequence defined by

$$L_{(k,l),0} = 2,\ L_{(k,l),1} = 1,\ L_{(k,l),n} = kL_{(k,l),n-1} + lL_{(k,l),n-2}\ (n \geq 2). \tag{3}$$

From (2) and (3), it readily follows that

$$F_{(k,l),n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} \tag{4}$$

and

$$L_{(k,l),n} = \frac{1 - 2\sigma_2}{\sigma_1 - \sigma_2}\sigma_1^n - \frac{1 - 2\sigma_1}{\sigma_1 - \sigma_2}\sigma_2^n, \tag{5}$$

where $\sigma_1 = \frac{k+\sqrt{k^2+4l}}{2}$ and $\sigma_2 = \frac{k-\sqrt{k^2+4l}}{2}$, respectively.

Definition 1.3. *The sequence*

$$\Delta^i \mathbf{F}_{k,l} = (\Delta^i F_{(k,l),0}, \Delta^i F_{(k,l),1}, \Delta^i F_{(k,l),2}, \ldots)^T,$$

defined inductively by $\Delta^i \mathbf{F}_{k,l} = \Delta(\Delta^{i-1}\mathbf{F}_{k,l})$, where $\Delta^0 \mathbf{F}_{k,l} = \mathbf{F}_{k,l}$, is called the ith difference sequence of $\mathbf{F}_{k,l}$. The sequence

$$\Delta^i \mathbf{L}_{k,l} = (\Delta^i L_{(k,l),0}, \Delta^i L_{(k,l),1}, \Delta^i L_{(k,l),2}, \ldots)^T,$$

defined inductively by $\Delta^i \mathbf{L}_{k,l} = \Delta(\Delta^{i-1}\mathbf{L}_{k,l})$*, where* $\Delta^0 \mathbf{L}_{k,l} = \mathbf{L}_{k,l}$*, is called the* i*th difference sequence of* $\mathbf{L}_{k,l}$*. For brevity,* $\Delta^i \mathbf{F}_{k,l}$ $(\Delta^i \mathbf{L}_{k,l})$ *and* $\Delta^i F_{(k,l),n}$ $(\Delta^i L_{(k,l),n})$ *will be denoted by* $\mathbf{F}_{k,l}^{(i)}$ $(\mathbf{L}_{k,l}^{(i)})$ *and* $F_{(k,l),n}^{(i)}$ $(L_{(k,l),n}^{(i)})$*, respectively.*

Both the Fibonacci and Lucas sequences are naturally connected to the Pascal matrix [3, 9], so we can easily guess that this would be the case for their difference sequences. It is thus enticing to investigate the relationships between the difference sequences of $\mathbf{F}_{k,l}$ and $\mathbf{L}_{k,l}$ and the Pascal matrix for a pair of positive real numbers k and l. The relationships allow us to obtain in-depth results for the structures of (k,l)-Fibonacci and (k,l)-Lucas difference sequences. This paper is organized as follows. In section 2, general notation and definitions are given. In section 3, the recurrence relations of the difference sequences of $\mathbf{F}_{k,l}$ and $\mathbf{L}_{k,l}$ are examined by applying a recurrence relation among entries in their difference matrices for a pair of positive real numbers k and l. In section 4, the difference matrices of the (k,l)-Fibonacci and (k,l)-Lucas sequences are represented by using the Pascal matrix, a Hankel matrix, and a Toeplitz matrix, which appear in problems entailing power and trigonometric moments [7].

2. Notation and definitions

The following notation and conventions are used throughout the manuscript:

- Infinite matrices have infinite number of rows i and columns j, with $i, j \in \{0, 1, \ldots\}$.
- The binomial coefficient ("i choose j") is denoted by $\binom{i}{j}$ with the convention that it equals 0 when $i < j$ or $j < 0$.
- $P = \left[\binom{i}{j}\right]$, $(i, j = 0, 1, \ldots)$ denotes the (infinite) *Pascal matrix*.
- Infinite real sequences $\{x_n\}$ are identified with $\mathbb{R}^\infty$ consisting of column vectors $\mathbf{x} = (x_0, x_1, x_2 \ldots)^T$ where $\mathbb{R}^\infty$ is the infinite dimensional real vector space.
- For a sequence $\mathbf{a} = (a_0, a_1, a_2, \ldots)^T$, $\Delta\mathbf{a} = (\Delta a_0, \Delta a_1, \Delta a_2, \ldots)^T$ denotes the difference sequence of $\mathbf{a}$ where $\Delta a_i = a_{i+1} - a_i$ for each $i = 0, 1, 2, \ldots$. $\Delta^k \mathbf{a} = (\Delta^k a_0, \Delta^k a_1, \Delta^k a_2, \ldots)^T$, $k = 0, 1, 2, \ldots$, denotes the kth difference sequence defined inductively by $\Delta^k \mathbf{a} = \Delta(\Delta^{k-1}\mathbf{a})$, where $\Delta^0 \mathbf{a} = \mathbf{a}$. For brevity, $\Delta^k \mathbf{a}$ and $\Delta^k a_n$ are denoted by $\mathbf{a}^{(k)}$ and $a_n^{(k)}$ for $n = 0, 1, 2, \ldots$.

The following appears in [6] without proof, so we present a simple proof for completeness and later discussion.

Lemma 2.1. *Let* k, n *be nonnegative integers. Then for* $\mathbf{a} = (a_0, a_1, a_2, \ldots)^T$ *in* $\mathbf{R}^\infty$*, we have*

$$a_n^{(k)} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{n+k-j}.$$

Proof. The proof proceeds by induction on k. If $k = 0$, then

$$a_n^{(0)} = (-1)^0 \binom{0}{0} a_n = a_n,$$

and we can commence the induction. Let $k \geq 1$. Then, since

$$a_n^{(k-1)} = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} a_{n+k-1-j}$$

and $a_{n+1}^{(k-1)} = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} a_{n+k-j}$ by the induction assumption, we have

$$\begin{aligned} a_n^{(k)} &= a_{n+1}^{(k-1)} - a_n^{(k-1)} \\ &= \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} a_{n+k-j} - \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} a_{n+k-1-j} \\ &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{n+k-j}, \end{aligned}$$

and the proof is complete. □

For a sequence $\mathbf{a} = (a_0, a_1, a_2, \ldots)^T \in \mathbf{R}^\infty$, let $\mathbb{M}^{\mathbf{a}}$ denote the difference matrix of $\mathbf{a}$ defined by

$$\mathbb{M}^{\mathbf{a}} = \begin{bmatrix} \mathbf{a}^T \\ \mathbf{a}^{(1)T} \\ \mathbf{a}^{(2)T} \\ \vdots \end{bmatrix} \tag{6}$$

and let $\mathbf{a}^\star = (a_0, a_0^{(1)}, a_0^{(2)}, \ldots)^T$, which is the first column of $\mathbb{M}^{\mathbf{a}}$, denote the dual sequence of $\mathbf{a}$ [8]. It is well known [1] that for each $i = 0, 1, 2, \ldots$,

$$a_n = a_0 \binom{n}{0} + a_0^{(1)} \binom{n}{1} + a_0^{(2)} \binom{n}{2} + \cdots + a_0^{(n)} \binom{n}{n}, \tag{7}$$

which is used in the proof of one of our main results. For a sequence $\mathbf{a}$, let $\mathbb{M}^{\mathbf{a}} = [m_{ij}]$. Then it follows from the construction of the difference matrix of $\mathbf{a}$ that the entries of $\mathbb{M}^{\mathbf{a}}$ satisfy the recurrence relation

$$m_{i+1,j} + m_{ij} = m_{i,j+1} \tag{8}$$

for $i, j = 0, 1, 2, \ldots$. As entries of $\mathbb{M}^{\mathbf{a}}$, the relative positions of the entries $m_{i+1,j}$, m_{ij}, $m_{i,j+1}$ in (8) is Γ-shaped. Thus, we call the relation the Γ-law, and we call a matrix a Γ-matrix if its entries satisfy the Γ-law. The difference matrix of a sequence is an example of Γ-matrices. The Γ-law in the difference matrix of a sequence plays a pivotal role in later discussions. We can guess that a Γ-matrix is related to a 7-matrix, which satisfies the 7-law [2]. Cheon et al. [2], showed that a 7-matrix can be represented by a Toeplitz matrix with a generalized Pascal matrix. This provides motivation to find a special matrix connected to the difference matrices. In the following lemma, the original recurrence relation of $\mathbf{F}_{k,l}$, and $\mathbf{L}_{k,l}$ is preserved in their difference sequences, which Falcon showed for $\mathbf{F}_{k,l}$ [6].

Lemma 2.2. *Let k and l be positive real numbers. For a pair of positive integers i and n, the following holds:*

(a) $F_{(k,l),n+1}^{(i)} = kF_{(k,l),n}^{(i)} + lF_{(k,l),n-1}^{(i)}$,

(b) $L^{(i)}_{(k,l),n+1} = kL^{(i)}_{(k,l),n} + lL^{(i)}_{(k,l),n-1}$.

Proof. For any positive real numbers α, β, k, and l, let $\mathbf{a} = (a_0, a_1, a_2, \ldots)^T$ denote a sequence defined by

$$a_0 = \alpha,\ a_1 = \beta,\ a_n = ka_{n-1} + la_{n-2},\ (n \geq 2). \tag{9}$$

For a nonnegative integer t, the recurrence relation of $\mathbf{a}$ is preserved in $\mathbf{a}^{(1)}$ since $a^{(1)}_t = a_{t+1} - a_t$, $a^{(1)}_{t+1} = la_t + ka_{t+1} - a_{t+1}$, and $a^{(1)}_{t+2} = la_{t+1} + k(la_t + ka_{t+1}) - la_t - ka_{t+1} = ka^{(1)}_{t+1} + la^{(1)}_t$. This tells us that we begin with the recurrence relation (9) of a sequence $\mathbf{a}$, which is preserved in the difference sequences of $\mathbf{a}$, and implies the results. □

We directly obtain the following result from Lemma 2.2 by means of the Γ-law. In the case of $l = 1$, the first clause is the result from Falcon [6].

Theorem 2.3. *For a pair of positive integers i and n, the following holds:*

(a) $F^{(i)}_{(k,l),n} = (k-1)F^{(i-1)}_{(k,l),n} + lF^{(i-1)}_{(k,l),n-1}$,

(b) $L^{(i)}_{(k,l),n} = (k-1)L^{(i-1)}_{(k,l),n} + lL^{(i-1)}_{(k,l),n-1}$ *where k and l are positive real numbers.*

Proof. We only prove the last clause because the other assertion can be proven similarly. It follows from the Γ-law and Lemma 2.2 that $L^{(i)}_{(k,l),n} + L^{(i-1)}_{(k,l),n} = L^{(i-1)}_{(k,l),n+1} = kL^{(i-1)}_{(k,l),n} + lL^{(i-1)}_{(k,l),n-1}$, which implies the last clause. □

We next consider the following difference matrices of $\mathbf{F}_{k,l}$ and $\mathbf{L}_{k,l}$.

Definition 2.4. *For positive real numbers k and l, let $\mathbf{F}_{k,l}$ and $\mathbf{L}_{k,l}$ be the (k,l)-Fibonacci and (k,l)-Lucas sequences, respectively. We call the two matrices $\mathbb{M}^{\mathbf{F}_{k,l}}$ and $\mathbb{M}^{\mathbf{L}_{k,l}}$ the difference matrices of $\mathbf{F}_{k,l}$ and $\mathbf{L}_{k,l}$, respectively.*

Our goal is to investigate the recurrence relations among entries in $\mathbb{M}^{\mathbf{F}_{k,l}}$ and $\mathbb{M}^{\mathbf{L}_{k,l}}$, extending the work by [6] by applying the Γ-law to the entries in $\mathbb{M}^{\mathbf{F}_{k,l}}$ and $\mathbb{M}^{\mathbf{L}_{k,l}}$. Falcon's result [6] are the recurrence relations and sums among entries in $\mathbb{M}^{\mathbf{F}_{k,1}}$. Finally, we show that $\mathbb{M}^{\mathbf{F}_{k,l}} = P^{-1}\mathbb{H}^{\mathbf{F}_{k,l}}$ and $\mathbb{M}^{\mathbf{L}_{k,l}} = P^{-1}\mathbb{H}^{\mathbf{L}_{k,l}}$ where $\mathbb{H}^{\mathbf{a}}$ is an infinite Hankel matrix with $\mathbf{a}^T$ as its first row for a sequence $\mathbf{a}$. This has direct consequences associated with the difference sequences of $\mathbf{F}_{k,l}$ and $\mathbf{L}_{k,l}$.

3. The difference matrices of $\mathbf{F}_{k,l}$ and $\mathbf{L}_{k,l}$

In this section, we show generalizations of the recurrence relations and generating functions of k-Fibonacci sequences from Falcon [6]. We begin with a simple recurrence relation for sums in $\mathbb{M}^{\mathbf{F}_{k,l}}$ and $\mathbb{M}^{\mathbf{L}_{k,l}}$, which is a direct consequence from the Γ-law.

Lemma 3.1. *For positive real numbers k and l, let $\mathbb{M}^{\mathbf{F}_{k,l}} = [f_{ij}]$ and $\mathbb{M}^{\mathbf{L}_{k,l}} = [l_{ij}]$ for $i, j = 0, 1, 2, \ldots$. Let n and t be nonnegative integers.*

(a) *If $t \geq 1$, then $\sum_{j=0}^{n} f_{tj} = f_{t-1,n+1} - f_{t-1,0}$ and $\sum_{j=0}^{n} l_{tj} = l_{t-1,n+1} - l_{t-1,0}$.*

(b) *If $t \geq 0$, then $\sum_{j=0}^{n} f_{t-j,j} = f_{t-n,n+1} - f_{t+1,0}$ and $\sum_{j=0}^{n} l_{t-j,j} = l_{t-n,n+1} - l_{t+1,0}$.*

Proof. (a) Let $t \geq 1$. By the Γ-law, we obtain $f_{tj} + f_{t-1,j} = f_{t-1,j+1}$ for $j = 0, 1, 2, \ldots, n$. Thus, $\sum_{j=0}^{n}(f_{tj} + f_{t-1,j}) = \sum_{j=0}^{n} f_{t-1,j+1}$, which implies that $\sum_{j=0}^{n} f_{tj} + f_{t-1,0} = f_{t-1,n+1}$. This proves the first case of part (a). For the case of $\sum_{j=0}^{n} l_{tj}$, the result is similarly obtained. (b) Using the Γ-law, the result can be proven similarly. □

The following two theorems are generalizations of Lemma 3.1.

Theorem 3.2. *For positive real numbers k and l, let $\mathbb{M}^{\mathbf{F}_{k,l}} = [f_{ij}]$ and $\mathbb{M}^{\mathbf{L}_{k,l}} = [l_{ij}]$ for $i, j = 0, 1, 2, \ldots$. Let n, p, and t be nonnegative integers with $t \geq 1$. Then we have*

(a) $\sum_{j=0}^{n}(\binom{p}{0} f_{tj} + \binom{p}{1} f_{t+1,j} + \binom{p}{2} f_{t+2,j} + \cdots + \binom{p}{p} f_{t+p,j}) = f_{t-1,n+p+1} - \sum_{j=0}^{p} \binom{p}{j} f_{t-1+j,0}$,

(b) $\sum_{j=0}^{n}(\binom{p}{0} l_{tj} + \binom{p}{1} l_{t+1,j} + \binom{p}{2} l_{t+2,j} + \cdots + \binom{p}{p} l_{t+p,j}) = l_{t-1,n+p+1} - \sum_{j=0}^{p} \binom{p}{j} l_{t-1+j,0}$.

Proof. (a) The proof proceeds by induction on p with $p \geq 0$. For $p = 0$, the result holds according to Lemma 3.1 (a), and we commence the induction. Let $p \geq 1$. Based on the induction hypothesis, we have

$$\begin{aligned}&\sum_{j=0}^{n}\left(\binom{p-1}{0} f_{tj} + \binom{p-1}{1} f_{t+1,j} + \cdots + \binom{p-1}{p-1} f_{t+p-1,j}\right)\\ &= f_{t-1,n+p} - \sum_{j=0}^{p-1}\binom{p-1}{j} f_{t-1+j,0}\end{aligned} \tag{10}$$

and

$$\begin{aligned}&\sum_{j=0}^{n}\left(\binom{p-1}{0} f_{t+1,j} + \binom{p-1}{1} f_{t+2,j} + \cdots + \binom{p-1}{p-1} f_{t+p,j}\right)\\ &= f_{t,n+p} - \sum_{j=0}^{p-1}\binom{p-1}{j} f_{t+j,0}\end{aligned} \tag{11}$$

Since $\binom{p-1}{j} + \binom{p-1}{j-1} = \binom{p}{j}$ for $j = 1, 2, \ldots, p-1$ and $f_{t-1,n+p} + f_{t,n+p} = f_{t-1,n+p+1}$ by the Γ-law, we obtain the desired result from (10) and (11). Clause (b) can be proven similarly. □

Theorem 3.3. *For positive real numbers k and l, let $\mathbb{M}^{\mathbf{F}_{k,l}} = [f_{ij}]$ and $\mathbb{M}^{\mathbf{L}_{k,l}} = [l_{ij}]$ for $i, j = 0, 1, 2, \ldots$. Then for nonnegative integers n, p, and t,*

(a) $\sum_{j=0}^{n}(\binom{p}{0} f_{t-j,j} + \binom{p}{1} f_{t-j+1,j} + \binom{p}{2} f_{t-j+2,j} + \cdots + \binom{p}{p} f_{t-j+p,j})$
$= f_{t-n,n+p+1} - \sum_{j=0}^{p} \binom{p}{j} f_{t+1+j,0}$,

(b) $\sum_{j=0}^{n}(\binom{p}{0} l_{t-j,j} + \binom{p}{1} l_{t-j+1,j} + \binom{p}{2} l_{t-j+2,j} + \cdots + \binom{p}{p} l_{t-j+p,j})$
$= l_{t-n,n+p+1} - \sum_{j=0}^{p} \binom{p}{j} l_{t+1+j,0}$.

Proof. The results can be readily proven in the same manner as the proof of Theorem 3.2. □

In the following theorem, we provide generating functions for rows in $\mathbb{M}^{\mathbf{F}_{k,l}}$ and $\mathbb{M}^{\mathbf{L}_{k,l}}$.

Theorem 3.4. *For positive real numbers k and l, let* $\mathbb{M}^{\mathbf{F}_{k,l}} = [f_{ij}]$ *and* $\mathbb{M}^{\mathbf{L}_{k,l}} = [l_{ij}]$ *for* $i,j = 0,1,2,\ldots$. *Let* $g^i_{\mathbf{F}}(x) = \sum_{j=0}^{\infty} f_{ij}x^j$ *and* $g^i_{\mathbf{L}}(x) = \sum_{j=0}^{\infty} l_{ij}x^j$ *for* $i = 0,1,2,\ldots$. *Then we have*

(a) $g^i_{\mathbf{F}}(x) = \frac{(\sigma_1-1)^i-(\sigma_2-1)^i+(\sigma_1(\sigma_1-1)^i-\sigma_2(\sigma_2-1)^i-k((\sigma_1-1)^i-(\sigma_2-1)^i))x}{(\sigma_1-\sigma_2)(1-kx-lx^2)}$,

(b) $g^i_{\mathbf{L}}(x) = \frac{s(\sigma_1-1)^i-t(\sigma_2-1)^i+(s\sigma_1(\sigma_1-1)^i-t\sigma_2(\sigma_2-1)^i-k(s(\sigma_1-1)^i-t(\sigma_2-1)^i))x}{(\sigma_1-\sigma_2)(1-kx-lx^2)}$

where $\sigma_1 = \frac{k+\sqrt{k^2+4l}}{2}$ *and* $\sigma_2 = \frac{k-\sqrt{k^2+4l}}{2}$ *with* $1-2\sigma_2 = s$ *and* $1-2\sigma_1 = t$.

Proof. (a) For each $i \geq 0$,

$$
\begin{aligned}
g^i_{\mathbf{F}}(x) &= f_{i0} + f_{i1}x + f_{i2}x^2 + f_{i3}x^3 \cdots + f_{in}x^n + \cdots, \\
kxg^i_{\mathbf{F}}(x) &= kf_{i0}x + kf_{i1}x^2 + kf_{i2}x^3 + \cdots + kf_{i-1,n}x^n + \cdots, \qquad (12)\\
lx^2g^i_{\mathbf{F}}(x) &= lf_{i0}x^2 + lf_{i1}x^3 + + \cdots + lf_{i-2,n}x^n + \cdots.
\end{aligned}
$$

From Lemmas 2.2 and 13, it follows that $g^i_{\mathbf{F}}(x) - kxg^i_{\mathbf{F}}(x) - lx^2g^i_{\mathbf{F}}(x) = f_{i0} + (f_{i1} - kf_{i0})x$, which implies that $g^i_{\mathbf{F}}(x) = \frac{f_{i0}+(f_{i1}-kf_{i0})x}{1-kx-lx^2}$. By Lemma 2.1 and (4), we obtain: For $i = 0,1,2,\ldots$,

$$f_{i0} = \frac{1}{\sigma_1-\sigma_2}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(\sigma_1^{i-j} - \sigma_2^{i-j}) = \frac{(\sigma_1-1)^i - (\sigma_2-1)^i}{\sigma_1-\sigma_2}.$$

By applying the Γ-law to f_{i1}, the result follows. Using Lemma 2.1 and (5), clause (b) can be proven similarly. □

By applying the Γ-law, we determine a recurrence relation for the consecutive three entries in a column of $\mathbb{M}^{\mathbf{F}_{k,l}}$ and $\mathbb{M}^{\mathbf{L}_{k,l}}$ as follows, which also generalizes the result of Falcon [6].

Lemma 3.5. *For positive real numbers k and l, let* $\mathbb{M}^{\mathbf{F}_{k,l}} = [f_{ij}]$ *and* $\mathbb{M}^{\mathbf{L}_{k,l}} = [l_{ij}]$ *for* $i,j = 0,1,2,\ldots$. *Then we have*

(a) $f_{i+2,j} = (l+k-1)f_{ij} + (k-2)f_{i+1,j}$.

(b) $l_{i+2,j} = (l+k-1)l_{ij} + (k-2)l_{i+1,j}$.

Proof. (a) Based on the Γ-law and Lemma 2.2, we obtain $f_{ij}+f_{i+1,j} = f_{i,j+1}$ and $lf_{ij} + kf_{i,j+1} = f_{i,j+1} + f_{i+1,j+1}$. Thus, $f_{i+2,j} = f_{i+1,j+1} - f_{i+1,j} = lf_{ij} + kf_{i,j+1} - f_{i,j+1} - f_{i+1,j} = (l+k-1)f_{ij} + (k-2)f_{i+1,j}$. Clause (b) can be proven similarly. □

As indicated in Lemma 3.5, a form of (k,l)-Fibonacci (Lucas) sequences results in $(l+k-1, k-2)$-Fibonacci (Lucas) sequences by constructing $\mathbb{M}^{\mathbf{F}_{k,l}}$ and $\mathbb{M}^{\mathbf{L}_{k,l}}$ from $\mathbf{F}_{k,l}$ and $\mathbf{L}_{k,l}$, respectively. Using Lemma 3.5, we provide generating functions for columns in $\mathbb{M}^{\mathbf{F}_{k,l}}$ and $\mathbb{M}^{\mathbf{L}_{k,l}}$ as follows.

Theorem 3.6. *For positive real numbers k and l, let* $\mathbb{M}^{\mathbf{F}_{k,l}} = [f_{ij}]$ *and* $\mathbb{M}^{\mathbf{L}_{k,l}} = [l_{ij}]$ *for* $i,j = 0,1,2,\ldots$. *Let* $g^j_{\mathbf{F}}(x) = \sum_{i=0}^{\infty} f_{ij}x^j$ *and* $g^j_{\mathbf{L}}(x) = \sum_{i=0}^{\infty} l_{ij}x^j$ *for* $j = 0,1,2,\ldots$. *Then we have:*

(a) $g^j_{\mathbf{F}}(x) = \frac{\sigma_1^j-\sigma_2^j+(\sigma_1^j(\sigma_1-1)-\sigma_2^j(\sigma_2-1)-(k-2)(\sigma_1^j-\sigma_2^j))x}{(\sigma_1-\sigma_2)(1-(k-2)x-(l+k-1)x^2)}$,

(b) $g^j_{\mathbf{L}}(x) = \frac{s\sigma_1^j-t\sigma_2^j+(s\sigma_1^j(\sigma_1-1)-t\sigma_2^j(\sigma_2-1)-(k-2)(s\sigma_1^j-t\sigma_2^j))x}{(\sigma_1-\sigma_2)(1-(k-2)x-(l+k-1)x^2)}$

where $\sigma_1 = \frac{k+\sqrt{k^2+4l}}{2}$ *and* $\sigma_2 = \frac{k-\sqrt{k^2+4l}}{2}$ *with* $1-2\sigma_2 = s$ *and* $1-2\sigma_1 = t$.

Proof. (a) For each $j \geq 0$,

$$\begin{aligned} g^j_{\mathbf{F}}(x) &= f_{0j} + f_{1j}x + f_{2j}x^2 + f_{3j}x^3 \cdots + f_{nj}x^n + \cdots , \\ (k-2)xg^j_{\mathbf{F}}(x) &= (k-2)f_{0j}x + (k-2)f_{1j}x^2 + (k-2)f_{2j}x^3 + \cdots \\ &\quad + (k-2)f_{n-1,j}x^n + \cdots , \\ (l+k-1)x^2 g^j_{\mathbf{F}}(x) &= (l+k-1)f_{0j}x^2 + (l+k-1)f_{1j}x^3 + + \cdots \\ &\quad + (l+k-1)f_{n-2,j}x^n + \cdots . \end{aligned} \tag{13}$$

From Lemma 3.5, it follows that $g^j_{\mathbf{F}}(x) - (k-2)xg^j_{\mathbf{F}}(x) - (l+k-1)x^2 g^j_{\mathbf{F}}(x) = f_{0j} + (f_{1j} - (k-2)f_{0j})x$, which implies that $g^j_{\mathbf{F}}(x) = \frac{f_{0j}+(f_{1j}-(k-2)f_{0j})x}{1-(k-2)x-(l+k-1)x^2}$. Thus, the result can be obtained from (4). Using (5), clause (b) can be proven similarly. □

4. The structure of $\mathbb{M}^{\mathbf{F}_{k,l}}$ and $\mathbb{M}^{\mathbf{L}_{k,l}}$

In this section, we investigate the structure of $\mathbb{M}^{\mathbf{F}_{k,l}}$ and $\mathbb{M}^{\mathbf{L}_{k,l}}$ and the relationship between these two matrices and the Pascal matrix using Hankel and Toeplitz matrices. For a sequence $\mathbf{c} = (c_0, c_1, c_2, \ldots)^T$, let $\mathbb{H}^{\mathbf{c}}$ denote the infinite Hankel matrix

$$\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} & c_n & \cdots \\ c_1 & c_2 & c_3 & \cdots & c_n & c_{n+1} & \cdots \\ c_2 & c_3 & c_4 & \cdots & c_{n+1} & c_{n+2} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-2} & c_{2n-1} & \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{2n-1} & c_{2n} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let $K_n = [k_{ij}]$, $(i, j = 0, 1, 2, \ldots, n)$ denote the $(n+1) \times (n+1)$ reversal matrix [7] defined by

$$k_{i,j} = \begin{cases} 1, & \text{if } j = n - i, \\ 0, & \text{otherwise.} \end{cases}$$

$K_n\mathbb{H}^{\mathbf{c}}_n$ and $\mathbb{H}^{\mathbf{c}}_n K_n$ are Toeplitz matrices [7], where $\mathbb{H}^{\mathbf{c}}_n$ is the leading $(n+1) \times (n+1)$ principal submatrix of $\mathbb{H}^{\mathbf{c}}$. The difference matrix of a sequence is naturally related to the Pascal matrix P with an infinite Hankel matrix.

Lemma 4.1. *Let P denote the Pascal matrix and let $\mathbf{a} = (a_0, a_1, a_2, \ldots)^T$. Then,*

$$\mathbb{M}^{\mathbf{a}} = P^{-1}\mathbb{H}^{\mathbf{a}} = DPD\mathbb{H}^{\mathbf{a}}$$

where $D = diag(1, -1, 1, -1, \ldots)$.

Proof. For each pair of nonnegative integers $i, j = 0, 1, 2, \ldots$, we have

$$(P\mathbb{M}^{\mathbf{a}})_{ij} = a_j\binom{i}{0} + a^{(1)}_j\binom{i}{1} + a^{(2)}_j\binom{i}{2} + \cdots + a^{(i)}_j\binom{i}{i} = a_{i+j}$$

by (7). Since $a_{i+j} = (\mathbb{H}^{\mathbf{a}})_{ij}$ and $P^{-1} = DPD$, the result follows. □

The following two results, shed more light on the combinatorial relationship between the difference sequences of $\mathbf{F}_{k,l}$ and $\mathbf{L}_{k,l}$, and are direct consequences of Lemma 4.1.

Theorem 4.2. *Let P denote the Pascal matrix. Then we have:*
(a) $\mathbb{M}^{\mathbf{F}_{k,l}} = P^{-1}\mathbb{H}^{\mathbf{F}_{k,l}}$,
(b) $\mathbb{M}^{\mathbf{L}_{k,l}} = P^{-1}\mathbb{H}^{\mathbf{L}_{k,l}}$.

Corollary 4.3. *Let P_n denote the $(n+1)\times(n+1)$ Pascal matrix for $n = 0, 1, 2, \ldots$. Then we have:*
(a) $\mathbb{M}_n^{\mathbf{F}_{k,l}} = P_n^{-1}\mathbb{H}_n^{\mathbf{F}_{k,l}} = P_n^{-1}K_n\mathbb{T}_n^{\mathbf{F}_{k,l}}$,
(b) $\mathbb{M}_n^{\mathbf{L}_{k,l}} = P_n^{-1}\mathbb{H}_n^{\mathbf{L}_{k,l}} = P_n^{-1}K_n\mathbb{T}_n^{\mathbf{L}_{k,l}}$
where for a sequence $\mathbf{c} = (c_0, c_1, c_2, \ldots)^T$, and $\mathbb{T}_n^{\mathbf{c}}$ is a Toeplitz matrix defined by

$$\mathbb{T}_n^{\mathbf{c}} = \begin{bmatrix} c_n & c_{n+1} & c_{n+2} & \cdots & c_{2n-1} & c_{2n} \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-2} & c_{2n-1} \\ c_{n-2} & c_{n-1} & c_n & \cdots & c_{2n-3} & c_{2n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_n & c_{n+1} \\ c_0 & c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix}$$

with $K_n\mathbb{H}_n^{\mathbf{c}} = \mathbb{T}_n^{\mathbf{c}}$.

The following result associates the difference sequences of the (k,l)-Fibonacci and (k,l)-Lucas sequences.

Corollary 4.4. *Let P_n denote the $(n+1)\times(n+1)$ Pascal matrix for $n = 0, 1, 2, \ldots$. Then we have:*
(a) $\mathbb{M}_n^{\mathbf{F}_{k,l}}\mathbb{M}_n^{\mathbf{L}_{k,l}} = P_n^{-1}\mathbb{H}_n^{\mathbf{F}_{k,l}}P_n^{-1}\mathbb{H}_n^{\mathbf{L}_{k,l}} = P_n^{-1}K_n\mathbb{T}_n^{\mathbf{F}_{k,l}}P_n^{-1}K_n\mathbb{T}_n^{\mathbf{L}_{k,l}}$,
(b) $\mathbb{M}_n^{\mathbf{L}_{k,l}}\mathbb{M}_n^{\mathbf{F}_{k,l}} = P_n^{-1}\mathbb{H}_n^{\mathbf{L}_{k,l}}P_n^{-1}\mathbb{H}_n^{\mathbf{F}_{k,l}} = P_n^{-1}K_n\mathbb{T}_n^{\mathbf{L}_{k,l}}P_n^{-1}K_n\mathbb{T}_n^{\mathbf{F}_{k,l}}$.

Proof. This directly follows from Corollary 4.3. □

We conclude this section with the following direct consequences from the previous three results.

- $P_n\mathbb{M}_n^{\mathbf{F}_{k,l}}$ and $P_n\mathbb{M}_n^{\mathbf{L}_{k,l}}$ are symmetric for $n = 0, 1, 2, \ldots$.
- For a pair of nonnegative integers i and j, $F_{(k,l),i+j} = \binom{i+t}{0}F_{(k,l),j-t} + \binom{i+t}{1}F^{(1)}_{(k,l),j-t} + \cdots + \binom{i+t}{i+t}F^{(i+t)}_{(k,l),j-t}$ $(t = 0, 1, \ldots, n-i)$.
- For a pair of nonnegative integers i and j, $L_{(k,l),i+j} = \binom{i+t}{0}L_{(k,l),j-t} + \binom{i+t}{1}L^{(1)}_{(k,l),j-t} + \cdots + \binom{i+t}{i+t}L^{(i+t)}_{(k,l),j-t}$ $(t = 0, 1, \ldots, n-i)$.
- For a pair of nonnegative integers i and j,

$$\sum_{z=0}^{n} F^{(i)}_{(k,l),z}L^{(z)}_{(k,l),j} = \sum_{z=0}^{n} S^{F_{(k,l),z}}_{t=0\to i}S^{L_{(k,l),j}}_{t=0\to z}$$

  where

$$S^{F_{(k,l),z}}_{t=0\to i} = \sum_{t=0}^{i}(-1)^{i+t}\binom{i}{t}F_{(k,l),z+t}$$

and

$$S^{L_{(k,l),j}}_{t=0\to z} = \sum_{t=0}^{z} (-1)^{z+t} \binom{z}{t} L_{(k,l),j+t}.$$

- For a pair of nonnegative integers i and j,

$$\sum_{z=0}^{n} L^{(i)}_{(k,l),z} F^{(z)}_{(k,l),j} = \sum_{z=0}^{n} S^{L_{(k,l),z}}_{t=0\to i} S^{F_{(k,l),j}}_{t=0\to z}$$

where

$$S^{L_{(k,l),z}}_{t=0\to i} = \sum_{t=0}^{i} (-1)^{i+t} \binom{i}{t} L_{(k,l),z+t}$$

and

$$S^{F_{(k,l),j}}_{t=0\to z} = \sum_{t=0}^{z} (-1)^{z+t} \binom{z}{t} F_{(k,l),j+t}.$$

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